Method for Obtaining Spectrally Efficient Orthogonal UWB Pulse Shapes

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Abstract—In this paper we present a method to obtain a set of orthogonal pulses to be used in Pulse Shape Modulation (PSM) for Ultra-Wide Band (UWB) communications. The pulses are built as linear combinations of Hermite functions which are shown to have unique advantageous features. Mathematical restrictions of orthogonality and spectral efficiency are introduced as guidelines to a fully explained search procedure to find the best set of pulses. Additionally, this procedure is adapted and used to find a single FCC-compliant pulse shape.

Keywords — Ultra-wide bandwidth, impulse radio, wireless, orthogonal modulation, orthogonal functions, Hermite functions, pulse shape modulation, pulse shaping, spectral compliance.

I. INTRODUCTION

Ultra-Wide Bandwidth communications (UWB) is receiving a growing interest from researchers addressing important developments for this kind of systems. In its original form, Impulse Radio, signals are impulsive, i.e., they are extremely short in time and therefore have a highly-spread spectrum. In order to convey data information, pulses are directly controlled by a pulse-modulation scheme which is of great importance to the overall system performance. The most common schemes are based in Pulse Position Modulation (PPM) [1], or in Pulse Amplitude Modulation (PAM).

A new modulation technique has been proposed by the authors making use of the shape of the pulse – Pulse Shape Modulation [2] (PSM). It was originally based on two waveforms orthogonal in time commonly used for modeling the pulse shape. In the same work, the concept was developed to include M orthogonal pulses, with a spectral restriction, in order to achieve M-ary orthogonal PSM, and a set of four orthogonal pulses was found.

The use of Hermite functions in UWB communications has been addressed in several other recent works, as in [3] and [4], which use a modified Hermite function, and [5] which brings to discussion the effect of the antennae on the transmitted pulses. The work of [6] shows that UWB antennae are undergoing evolution. As long as the effect of the channel on the transmitted signal varies with the antenna type, independently developing the UWB waveforms is still a good approach. Moreover, after choosing an advantageous pulse waveform, one can choose finding an adequate antenna or previously modify the signal to be send, in a pre-warping approach.

The present work describes in detail the method used for finding higher order sets of orthogonal pulses for PSM, which is a consolidation of the general procedure introduced in [2]. It is also shown how this flexible method can be adapted to obtain pulses with specific spectral criteria, through the important example of designing a pulse compliant to the FCC spectral mask. The set of 4 orthogonal pulses obtained was used [7] in quaternary PSM schemes compared to other PSM and PPM schemes, showing good overall performance. Due to its good performance and spectral efficiency, we expect it will work well also in more elaborate channels, e.g. multipath channels.

In the next section we review the main types of UWB modulation expliciting their differences in terms of their expressions for the modulated pulse streams and theoretical performances. In section III, the Hermite functions are introduced and their utility is explained using the concept of Hermite spaces. Section IV describes in detail the method for obtaining orthogonal pulse shapes with some desired properties, based on Hermite functions. In section V, as an example of how the method can be adapted to obtain pulses in conformity to very specific spectral requirements, we describe the design of a pulse complying to the FCC requirements for indoor systems. Finally, some conclusions are presented in section VI.

II. MODULATION TECHNIQUES

A. Pulse Amplitude Modulation

In Pulse Amplitude Modulation (PAM), a given waveform is sent with different amplitudes corresponding to different data being transmitted. The basic PAM-modulated signal composed by a stream of modulated pulses is given by

\[ x_A(t) = \sum_j A_j w(t - jT_f) \]  \hspace{1cm} (1)

where \( w \) is the pulse waveform, \( A_j \) is the amplitude of the \( j \)-th pulse corresponding to the symbol represented and \( T_f \) is the frame duration or pulse repetition time.
In binary modulation we may have the optimum antipodal case by making \( A_j = 1 \) representing bit 0 and \( A_j = -1 \) representing bit 1. In this case, detection may be performed by a single correlator using as template signal a normalized-energy pulse \( w(t) \), which will result in correlation values equal to 1 or -1.

### B. Pulse Position Modulation

In Pulse Position Modulation (PPM), each data symbol is represented by a particular delay in the transmitted pulse. The modulated pulse stream becomes

\[
x_P(t) = \Sigma_j w(t - jT_f - \tau_j)
\]

where \( \tau_j \) is the delay of the \( j \)-th pulse corresponding to the symbol represented.

In a binary modulation we have typically \( \tau_0 = 0 \) representing bit 0 and \( \tau_1 = \delta \) representing bit 1, where \( \delta \) is a time delay. Reception of a PPM binary signal can be made using a single correlator [1] with a a special template waveform obtained from the sum of the two possible waveforms, one of them inverted and delayed by \( \delta \), optimized for minimum cross-correlation.

\[
v(t) = w(t) - w(t - \delta)
\]

### C. Pulse Shape Modulation

Pulse Shape Modulation (PSM) was proposed in [2], and also independently in [3]. Both works make use of orthogonal Hermite functions. The work described in [3] uses a modulated version of Hermite functions, generating pulses whose spectra are clearly distinct, what may be a problem when we have bandwidth restrictions. The work described in [2] uses optimized combinations of Hermite functions in order to obtain pulses occupying roughly the same spectra, therefore achieving best spectral efficiency. Throughout the rest of this paper we refer only to the PSM modulation proposed in [2].

The original idea of PSM consists of an orthogonal binary modulation where two orthogonal waveforms are used to represent the data bits. The reception, based on signal correlation, benefits from the time-orthogonality between the waveforms used. The resulting modulated pulse stream is given by

\[
x_S(t) = \Sigma_j w_j(t - jT_f)
\]

where \( w_j(t) \) is the waveform associated with the \( j \)-th pulse according to the data.

For \( M \)-ary orthogonal PSM schemes we need to use sets of pulse waveforms which are all orthogonal to each other. This can be obtained from sets of orthogonal functions. The modulation proposed in [2] uses pulses obtained from the linear combination of a given number of orthogonal Hermite functions using the concept of linear signal spaces. Among many possibilities, a search procedure selected the optimum combination that result in a set of pulses occupying the most similar frequency bands. The method used for obtaining such pulses is detailed in this paper.

### III. Orthogonal Hermite Pulses

#### A. Hermite Functions

Sets of orthogonal functions arise from the solution of some partial differential equations, Hermite functions [8] [9] are one example. As it occurs with Fourier series, they can be used to expand some functions in terms of the base of orthogonal functions, which is possible for all square-integrable (or finite-energy) functions. This means that these functions form a basis of the space of finite-energy functions, referred to as \( L_2 \). This is a linear signal space [10], i.e., a collection of signals \( x(t) \) such that if \( x_1(t) \) and \( x_2(t) \) belong to the collection, then any linear combination \( c_1 x_1(t) + c_2 x_2(t) \) also belongs to it. In \( L_2 \) we also have that \( \|x\|^2 = \langle x, x \rangle = \int_t |x(t)|^2 dt \) has finite value.

We may expect that the frequency content of signals based on some set of Hermite functions will be the minimum necessary as a consequence of the maximal concentration in time and frequency of Hermite spaces (c.f. section III-B). These issues, along with the orthogonality of Hermite functions, constitute the motivation to use these functions. Figure 1 shows some of the first Hermite functions, given by

\[
\psi_n(t) = \frac{H_n(t)e^{-t^2/2}}{\sqrt{2^n n! \sqrt{\pi}}},
\]

where \( H_n(t) \) are the Hermite polynomials, recursively obtained by the formulas

\[
H_0(t) = 1;
\]
\[
H_1(t) = 2t;
\]
\[
H_{n+1}(t) = 2tH_n(t) - 2nH_{n-1}(t) \quad (n \geq 1).
\]

![First Three Hermite Functions](image)

A special feature that adds treatability to signal modelling through Hermite functions is the fact that they are autofunctions of the Fourier transform, which means

\[
\Psi_n(\omega) = F\{\psi_n(t)\} = j^n \psi_n(\omega)
\]

\( \Psi_n(\omega) \) is the waveform associated with the \( j \)-th pulse according to the data.

For \( M \)-ary orthogonal PSM schemes we need to use sets of pulse waveforms which are all orthogonal to each other. This can be obtained from sets of orthogonal functions. The modulation proposed in [2] uses pulses obtained from the linear combination of a given number of orthogonal Hermite functions using the concept of linear signal spaces. Among many possibilities, a search procedure selected the optimum combination that result in a set of pulses occupying the most similar frequency bands. The method used for obtaining such pulses is detailed in this paper.
B. Hermite Spaces

We can select finite subspaces from this infinite-dimension space \( L_2 \) by selecting a finite number of Hermite functions. The \( N \)-dimensional Hermite space is thus defined as the space spanned by the first \( N \) Hermite functions [10]

\[
\mathcal{H}_N = \text{span}\{\psi_n(t)\}_{n=0}^{N-1}
\]

The time-frequency analysis theory [10] proves that the Hermite spaces are maximally concentrated both in time and frequency. This allows getting the shortest effective duration waveforms without having a frequency content higher than necessary. Besides avoiding further difficulties for the signal generation, this guarantees optimal use of the communications resources measured by the duration-bandwidth product.

C. Hermite Pulses

Hermite pulses of \( N \)-th order are defined [2] as any signal waveform belonging to the \( N \)-th dimension Hermite space. These pulses can thus be expressed as

\[
w_{HN}(t) = g_{N-1}(t) \exp\left(-\frac{t^2}{2}\right)
\]

where \( g_{N-1}(t) \) is an \( (N-1) \)-degree polynomial. Alternatively, Hermite pulses can be expressed as

\[
w_{HN}(t) = \vec{\psi}_N(t) w_{HN}
\]

where

\[
\vec{\psi}_N(t) = [\psi_0(t) \psi_1(t) \ldots \psi_{N-1}(t)]
\]

and

\[
w_{HN} = [h_0 \ h_1 \ldots \ h_{N-1}]^T
\]

i.e., \( \vec{\psi}_N(t) \) is a vector composed by the first \( N \) Hermite functions and \( w_{HN} \) is the projection vector, a real column vector relating the considered pulse to the \( N \) first Hermite functions. This way, each signal of the Hermite space \( \mathcal{H}_N \) is expressed through its orthonormal basis.

IV. SEARCHING FOR ORTHOGONAL SETS

We wish to find a set of \( M \) \( N \)-th order Hermite pulses, \( M \leq N \), orthogonal to each other, with no DC component, and subject to some band restriction. This set can be represented by a matrix \( N \times M \) formed by the projection vectors \( w_{HN} \) from each pulse

\[
A = [ w_{HN,1} \ w_{HN,2} \ldots \ w_{HN,M} ]
\]

where the second index designates each particular pulse.

The requirement of absence of DC component eliminates the trivial solution of using exactly the Hermite functions as the set of desired pulses, posing a linear restriction to the vector components \( w_{HN} \) and, therefore, to the lines of matrix \( A \), leading to

\[
M \leq N - 1
\]

A. General Procedure to Obtain a Set of \( M \) Orthogonal Pulses

At this point we should search within the \( M \)-dimensional vector space represented by the columns of \( A \). In some sense, we are looking for an orthogonal basis of this space, where the basis vectors represent the pulses we want, to which we impose the following restrictions:

- to be orthogonal;
- to have zero DC component;
- to occupy the most possibly similar frequency bands.

Band restrictions are by now generalized by the last restriction. We therefore developed the following procedure, translating the above statements into mathematical restrictions:

1) Given \( M \), make (from equation (13))

\[
N = M + 1
\]

2) Determine the generic form of the projection vector and then build the equation expressing the pulse by the functions of the basis

\[
w_{HN}(t) = h_0\psi_0(t) + h_1\psi_1(t) + \ldots + h_{N-1}\psi_{N-1}(t)
\]

(For the purpose of clarity, whenever possible, we omit the index referring to the pulse.)

3) Apply the zero-DC restriction given by

\[
\int_{-\infty}^{\infty} w_{HN}(t) dt = 0
\]

a) Express the algebraic relation between elements of \( w_{HN} \)

\[
h_0 = f_{0DC}(h_1, \ldots h_{N-1})
\]

where \( f_{0DC} \) is a linear function resulting from (16).

b) Apply the same relation to \( A \), replacing its first line by a linear combination of the others. This reduces the number of variables in \( A \) from \( MN \) to \( M(N-1) \) (or to \( M^2 \), according to equation (14)).

4) Build the normalized orthogonality equations

\[
A^T A = I_M
\]

which consist of \( M(M+1)/2 \) equations with \( M^2 \) variables.

5) Obtain the generic expression for the mean central frequency and the frequency spreading:

a) Take the Fourier Transform of \( w_{HN} \)

\[
W_{HN}(\omega) = \mathcal{F}\{w_{HN}(t)\} = h_0\Psi_0(\omega) + h_1\Psi_1(\omega) + \ldots + h_{N-1}\Psi_{N-1}(\omega)
\]

b) Find the expression of the energy spectral density of a pulse given by

\[
E_w(\omega) = |W_{HN}|^2 = W_{HN} \cdot W_{HN}^*
\]
c) Write the expressions of mean central frequency and frequency spreading defined as
\[
\omega_0 = \frac{\int_0^\infty \omega |E_w(\omega)|^2 d\omega}{\int_0^\infty |E_w(\omega)|^2 d\omega} = 2 \int_0^\infty \omega |E_w(\omega)|^2 d\omega
\]
and
\[
\sigma_\omega^2 = \frac{\int_0^\infty (\omega - \omega_0)^2 |E_w(\omega)|^2 d\omega}{\int_0^\infty |E_w(\omega)|^2 d\omega} = 2 \int_0^\infty (\omega - \omega_0)^2 |E_w(\omega)|^2 d\omega
\]
respectively. Note that equations (19) – (22) are functions of \(h_0, h_1, ... h_{N-1}\).

d) Apply equation (17) to eliminate \(h_0\) from equations (21) and (22)
\[
\omega_0 = f_{\omega_0}(h_1, ..., h_{N-1})
\]
\[
\sigma_\omega^2 = f_{\sigma_\omega}(h_1, ..., h_{N-1})
\]
where \(f_{\omega_0}\) and \(f_{\sigma_\omega}\) have the same expressions of equations (21) and (22) but in terms of \(h_1, ..., h_{N-1}\) only.

e) Find the functional to be minimized.

a) Make
\[
\omega_1 = \omega_0 - \alpha \sigma_\omega
\]
and
\[
\omega_2 = \omega_0 + \alpha \sigma_\omega
\]
with a suggested value of \(\alpha = 1.3\) (this was empirically determined to fit best \(\omega_1\) and \(\omega_2\) to the cutoff frequencies of -3dB).

b) Determine
\[
G = \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} \left[ (\omega_{1,i} - \omega_{1,j})^2 + (\omega_{2,i} - \omega_{2,j})^2 \right]
\]
where \(i\) and \(j\) correspond to the pulses being considered.

7) Do an extensive search over all sets that are solutions to equation (18) looking for the one which minimizes \(G\) in equation (26).

In order to make this search feasible, we have developed an auxiliary procedure that uses orthogonal square matrices for which there are well known generating methods. The procedure is based on a recombined \(M \times M\) matrix \(A_R\):
\[
A_R = \begin{bmatrix}
  h_{1,1} & h_{1,2} & \cdots & h_{1,M} \\
h_{2,1} & h_{2,2} & \cdots & h_{2,M} \\
  \vdots & \vdots & \ddots & \vdots \\
h_{M,1} & h_{M,2} & \cdots & h_{M,M}
\end{bmatrix}
\]
(27)
which is the matrix \(A\) without the first line, being related to it through a combination matrix \(C\) due to the 0-DC restriction
\[
A = C A_R
\]
(28)
The procedure to search through the solutions of (18) is:
1) Find the (numerical) combination matrix \(C\), such that equation (28) is satisfied.
2) Make \(C_2 = C^T C\).
3) Decompose \(C_2\) in its eigenvalues and eigenvectors, finding \(V\) and \(\Lambda\) such that:
\[
C_2 = V^T \Lambda V
\]
(29)
Repeat steps 4 – 7 below for a suitable number of orthogonal matrices.
4) Generate an orthogonal matrix \(\tilde{A}\).
5) Find
\[
A_R = V^{-1}(\Lambda^{1/2})^{-1} \tilde{A}
\]
(30)
(We recall that \(V^{-1} = V^T\) and \(\Lambda\) is simply a diagonal matrix.)
6) Use equation (28) to get \(A\).
7) Using the elements of \(A\), calculate \(G\) (using equations (23) – (26)).

The whole procedure was applied to find a set of 4 orthogonal Hermite pulses. The pulses found are shown in figure 2, and corresponds to the following vectors:
\[
\begin{align*}
W_{1h_5} &= \begin{bmatrix}
  -4 \sqrt{4 + 2\sqrt{2}} \overline{4} \\
  0 \\
  4 \sqrt{4 + 2\sqrt{2}} \overline{4}
\end{bmatrix}^T \\
W_{2h_5} &= \begin{bmatrix}
  -4 \sqrt{4 + 2\sqrt{2}} \overline{4} \\
  0 \\
  4 \sqrt{4 + 2\sqrt{2}} \overline{4}
\end{bmatrix}^T \\
W_{3h_5} &= \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}^T \\
W_{4h_5} &= \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}^T
\end{align*}
\]
(31)

Fig. 2. Four Orthogonal Hermite Pulses.

The spectra of these pulses can be seen in figure 3. Two pairs of pulses have identical spectra, due to being two of them the time-inverted versions of the other two, as it can be seen in figure 2.

V. DESIGNING PULSES UNDER SPECIFIC SPECTRAL REQUIREMENTS

In some situations, it is desirable for an UWB system to conform to specific spectral restrictions. The method of the
previous section can be easily adapted to meet such restrictions. As an example, we show how the search procedure can be modified to find a single pulse waveform respecting the FCC regulation for indoor systems [11].

The FCC regulation allows a maximum spectral density within a window ranging from 3.1 to 10.6 GHz, at which points the allowed density falls 10 dB. There is also a thin band to be avoided, between 0.96 and 1.61 GHz, with a 34 dB lower limit to protect GPS systems. Therefore, besides respecting these limits, an UWB system should use efficiently the main spectral window in order to maximize its total allowable power.

With this in mind, we state that the Hermite space in which we will search must permit to obtain pulse spectra whose \(-10\) dB points keep a relation close to that of the FCC window:

\[
\text{FCC}_{-10dB} = 3.1/10.6 \approx 0.292
\]  (32)

which can be viewed as an alternative measure of the pulses bandwidth.

As we are looking for a single waveform, we have \(M = 1\), what implies that \(N \geq 2\). The value of \(N = 2\) clearly does not satisfy and is promptly discarded. Making a fast search over third-order pulses we realized that they never have \(-10\) dB relations greater than 0.201. So we must try \(N = 4\).

After finding the adequate value of \(N = 4\), we follow the procedure from steps 1 to 4. In this case, \(A\) reduces to a 4 \(\times\) 1 column-matrix with \(A^T A = 1\), leading to the normalization equation \(h_1^2 + \frac{2}{3} h_2^2 + h_3^2 = 1\). The following steps were modified to pursue our intended objective:

5) Obtain the pulse’s spectral energy density, either determining its literal expression or by numerical computation with a convenient frequency grid.

6) a) Find, on the spectral density curve, the points of \(-10\)dB (above and below the peak frequency) and \(-34\)dB (below the peak frequency). Namely, in ascending order:

\[
\omega_{-34dB_1}; \omega_{-34dB_2}; \omega_{-10dB_1}; \omega_{-10dB_2}.
\]

b) Calculate

\[
r_{-34dB_1} = \frac{\omega_{-34dB_1}}{\omega_{-10dB_2}};
\]

\[
r_{-34dB_2} = \frac{\omega_{-34dB_2}}{\omega_{-10dB_2}};
\]

\[
r_{-10dB} = \frac{\omega_{-10dB_1}}{\omega_{-10dB_2}};
\]

\[c) \text{Obtain}
G = \text{Max}\{\frac{r_{-34dB_1}}{\text{FCC}_{-34dB_1}} - 1; \frac{\text{FCC}_{-34dB_2}}{r_{-34dB_2}} - 1; \frac{\text{FCC}_{-10dB}}{r_{-10dB}} - 1\}; \]

where, similarly to \(\text{FCC}_{-10dB}\) defined at equation (32), we have

\[
\text{FCC}_{-34dB_1} = 0.96/10.6; \quad (37)
\]

\[
\text{FCC}_{-34dB_2} = 1.61/10.6. \quad (38)
\]

To scan the solution space, defined by the vector \([h_1 \ h_2 \ h_3]^T\), it is convenient to make

\[
h_1 = \sin \theta_1 \sin \theta_2; \quad (39)
\]

\[
h_2 = \frac{\sqrt{6}}{3} \cos \theta_1; \quad (40)
\]

\[
h_3 = \sin \theta_1 \cos \theta_2; \quad (41)
\]

with \(\frac{\pi}{4} \leq \theta_1 \leq \pi; \quad 0 \leq \theta_2 \leq \pi\).

An optimum pulse was found with \(\theta_1 = \frac{\pi}{2}\) and \(\theta_2 = \frac{11\pi}{15}\), corresponding to

\[
h_0 = h_2 = 0; \quad (42)
\]

\[
h_1 \approx 0.743; \quad (43)
\]

\[
h_3 \approx -0.669; \quad (44)
\]

what results in the following time equation:

\[
w_{\text{FCC}_{h_1}}(t) \approx (-0.5804 t^6 - 0.0811 t) e^{-t^2/2} \quad (45)
\]

An inverted version of this pulse is plotted in figure 4 and its spectral energy density, over the FCC mask, is shown in figure 5.

VI. CONCLUSIONS

In this work we presented a method to obtain PSM modulation with orthogonal pulse waveforms occupying the same frequency band, an important feature to make efficient use of the available spectrum. The method was suitably adapted for pulse shaping, originating a special pulse waveform fully compliant to the FCC spectral restrictions.

The orthogonality, the polarity independence and the time-frequency concentration of the pulses in our PSM scheme are promising for performing in environments where there is uncertainty with respect to the state of the received signal, such as channels with multipath and jitter.
REFERENCES
