DYNAMIC OUTPUT FEEDBACK COMPENSATOR FOR TIME-STAMPED NETWORKED CONTROL SYSTEMS

VITOR M. MORAES*, EUGÊNIO B. CASTELAN*, UBIRAJARA F. MORENO*

∗Grupo de Controle de Sistemas Mecatrônicos - CSM
Departamento de Automação e Sistemas - DAS
Universidade Federal de Santa Catarina - UFSC
Florianópolis, SC, Brasil

Emails: rattus@das.ufsc.br, eugenio@das.ufsc.br, moreno@das.ufsc.br

Abstract— We propose a dynamic output-feedback compensator for time-stamped Networked Control Systems (NCS). The design methodology is based on the use of a polytopic with additive norm-bounded representation for the NCS. The dynamics of the proposed compensator depends on the delays induced by the network, which are supposed not to be greater than the sampling period and possible to compute from the information provided by time-stamped messages. The closed-loop stability and some time-performance are guaranteed from the use of a parameter dependent candidate Lyapunov function. The synthesis of the compensator gains are described in terms of Linear Matrix Inequalities (LMIs). A numerical example and simulations of the NCS are provided to show the applicability of the proposed compensator.

Keywords— Networked control systems, dynamic output feedback, time varying delays, time-stamped messages, parameter dependent Lyapunov function.

1 Introduction

Networked Control Systems (NCS) may have their behavior disturbed due to communication constraints, such as variable delays, rendering more difficult to analyze and design controllers that guarantee stability and some desired performance (Tang and Yu, 2007).

The increasing use of NCS has lead to a corresponding advance in the related area of research (Baiillicul and Antsaklis, 2007; Ge et al., 2007; Hespanha et al., 2007). In particular, many results in this area use the Lyapunov theory for establishing stability conditions, used for analysis and control synthesis (see, for instance, Cloosterman et al. (2010), Hetel et al. (2007), and references therein). Additionally, robust control techniques can be used to describe the system model, such as convex polytopes and norm-bounded uncertainties, and constraints, such as guaranteed-cost and $H_{\infty}$ characterization, in order to reduce conservativeness. Many of these works found in the literature are related to the design of state-feedback compensators, which in practice are not always possible to be implemented.

In this context, recently more attention has been dedicated concerning output feedback control synthesis for NCSs. Some examples are the design techniques based on the use of state-observers (Montestruque and Antsaklis, 2002; Wen-ying et al., 2009) or single output matrix gain (Yoo et al., 2010; Dritsas and Tzes, 2007). Also, dynamic output feedback has been proposed (Hao and Zhao, 2010; Shi and Yu, 2011; Rasool et al., 2012), mostly based on discrete-time system descriptions. However, in practical applications, continuous-time plants with discrete-time controllers are used. Also, random time-varying delays, different from the sample time multiples, may happen, rendering a difficult aspect to analyze. In Heemels and van de Wouw (2010) it is pointed out the difficulty of the design of output-based dynamic discrete-time controllers that result in stable closed-loop for this kind of NCSs. There are some techniques for stability analysis, see Donkers et al. (2009) for example, but we can notice a lack of synthesis approaches in the literature.

The aim of the present paper is to propose a method for the synthesis of a dynamic output feedback compensator (DOFC) for NCS, based on the exact discretization of the closed-loop linear system. Time-stamped messages are considered to be sent over the NCS, allowing the proposition and use of a compensator that is dependent on the network induced time-delay parameter and that has the same order of the system to be controlled. In one side, the proposed results generalizes the result presented in Moraes et al. (2011), in the sense that the proposed compensator can have all its matrices dependent of the delay. On the other hand, it is also shown that, by considering a particular structure of this compensator, the originally Bilinear Matrix Inequality (BMI) synthesis condition becomes a Linear Matrix Inequality (LMI) condition.

The paper is organized as follows. Section 2 describes the NCS configuration and the used discrete-time model. Also in section 2, the desired compensator structure is presented. In section 3, we present the preliminary concepts and a stability result. The synthesis results are proposed in section 4 and a simple numerical example with simulation is presented in section 5. A brief conclusion ends the paper.

Notation: $A^T$ denotes the transpose of $A$. In general,
2 Problem Formulation

Consider a system as shown in Fig. 1, where the open-loop system, or plant, is linear time-invariant and described by:

\[
\begin{aligned}
\dot{x}(t) &= Mx(t) + Nu(t) \\
y(t) &= Cx(t)
\end{aligned}
\]  

(1)

with \(x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p, M \in \mathbb{R}^{n \times n}, N \in \mathbb{R}^{n \times m}\) and \(C \in \mathbb{R}^{p \times n}\). Communication between the control system components is made by messages sent through the shared network environment.

![Networked Control System](image)

\[\text{Figure 1: Networked Control System}\]

The time-driven sensor samples the system output at regular time intervals \(T\), sending acquired information to the compensator through a time-stamped message. The actuator is event-driven and acts as a zero order hold, updating the control signal applied to the plant when new messages from the compensator are received. Also, the actuator sends a time-stamp message to the compensator, informing the instant when the control signal applied to the plant has changed. The controller is event-driven, sending to the actuator a new control signal as soon as a new message arrives from the sensor (Moraes et al., 2010; Hetel et al., 2007). Moreover, we consider the clock synchronization among the control system components.

Deterministic protocols are considered, allowing a correct scheduling of the network access, in order to guarantee the message deadlines. However, delay can occur between the instants of sample and actuation, occasionally leading to performance degradation or even instability (Zhang and Yu, 2008). Such a delay can vary due to the wait caused by other nodes trying to access the network. Assuming a maximum deadline equal to the sampling period, we have \(0 < \tau_{\text{min}} \leq \tau_k \leq \tau_{\text{max}} \leq T\), where \(\tau_k\) represents time-varying delay, associated to the discrete-time instant \(k\). For simplicity, it is assumed the use of time-stamp messages, allowing to implement a control law dependent on the parameter \(\tau_k\). However, the proposed technique can also be adapted to the case where only a delay estimation is available, using an approach similar to the one used in Hetel et al. (2011).

2.1 Polytopic Representation

The discrete-time representation of system (1) is obtained by considering that the time-delay \(\tau_k\) is applied to the control input, that is, for \(t \in [kT, (k+1)T)\):

\[
u(t) = \begin{cases} 
    u_{k-1}, & t \in [kT, kT + \tau_k) \\
    u_k, & t \in [kT + \tau_k, (k+1)T)
\end{cases}
\]

(2)

It leads to the following representation (Åström and Wittenmark, 1997):

\[
x_{k+1} = Ax_k + \Gamma_1(\tau_k)u_{k-1} + \Gamma_0(\tau_k)u_k \\
y_k = Cx_k
\]

(3)

where \(A = e^{MT}, \Gamma_0(\tau_k) = \int_0^{T-\tau_k} e^{Ms}ds N, \Gamma_1(\tau_k) = \int_0^T e^{Ms}ds N = B - \Gamma_0(\tau_k)\), with \(B = \int_0^T e^{Ms}ds N\).

The time-delay dependent matrix \(\Gamma_0(\tau_k)\) is expressed by a polytopic form, with additive norm-bounded residual uncertainty, based on a Taylor series decomposition approach (see Hetel et al. 2007) for more details:

\[
\Gamma_0(\tau_k) = \sum_{i=1}^{h+1} \mu_i(\tau_k)\Gamma_{0i} + \Delta(\tau_k)
\]

where the vertices of the polytope are given by \(\Gamma_{0i} = \begin{bmatrix} M_{i-1} & \cdots & M_0 & I \end{bmatrix}\phi_i N\), for \(i = 1, \ldots, h+1\), and:

\[
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_{h+1}
\end{bmatrix} =
\begin{bmatrix}
\alpha^h I \\
\alpha^{h-1} I \\
\vdots \\
I
\end{bmatrix}
\]

with \(\alpha = T - \tau_{\text{max}}\) and \(\pi = T - \tau_{\text{min}}\). The weighting factors \(\mu_i(\tau_k)\) are obtained from the linear system:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha & \alpha & \cdots & \alpha \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^h & \alpha^{h-1} & \cdots & \alpha^0
\end{bmatrix}
\begin{bmatrix}
\mu_{1k} \\
\mu_{2k} \\
\vdots \\
\mu_{(h+1)k}
\end{bmatrix} = \begin{bmatrix} 1 \\
\alpha_k \\
\vdots \\
\alpha_k^h
\end{bmatrix}
\]

(4)

Developments corresponding to \(\Gamma_1(\tau_k)\) can be obtained from the relation \(\Gamma_1(\tau_k) = B - \Gamma_0(\tau_k)\).
2.2 Closed-loop system:

According to the previous description, the problem considered in this article consists of the synthesis of a parameter dependent dynamic output feedback compensator (DOFC) that guarantees stability and some time performance of the closed-loop system.

The following n-order DOFC is proposed, which has a similar structure as the one used in Moraes et al. (2011), but has all of its matrices dependent on the parameter $r_k$:

$$
\zeta_{k+1} = A_c(\tau_k)\zeta_k + B_c(\tau_k)y_k + [F_1(\tau_k) \ F_0(\tau_k)]u_k
$$

$$
u_{k+1} = C_c(\tau_k)\zeta_k + D_c(\tau_k)y_k + [K_1(\tau_k) \ K_0(\tau_k)]v_k
$$

(5)

where $v_k = [u_{k-1}^T ~ u_k^T]^T$, and

$$
[A_c(\tau_k) ~ B_c(\tau_k) ~ F_1(\tau_k) ~ F_0(\tau_k)] =
\sum_{i=1}^{h+1} \mu_i(\tau_k) [A_{ci} ~ B_{ci} ~ F_{1i} ~ F_{0i}]
$$

$$
[C_c(\tau_k) ~ D_c(\tau_k) ~ K_1(\tau_k) ~ K_0(\tau_k)] =
\sum_{i=1}^{h+1} \mu_i(\tau_k) [C_{ci} ~ D_{ci} ~ K_{1i} ~ K_{0i}]
$$

By defining the auxiliary variable $\tau_k$ such that $x_k = \tau_k - B u_k$, with $B \in \mathbb{R}^{n \times m}$, and considering the augmented vector $\zeta_k = [\tau_k^T \ u_{k-1}^T \ u_k^T]^T \in \mathbb{R}^{l}$, $l = 2(n + m)$, a closed-loop representation is obtained:

$$
z_{k+1} = H(\tau_k)z_k + E\Delta(\tau_k)Dz_k
$$

(6)

where $E' = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & -I \end{bmatrix}$, $H(\tau_k) = \sum_{i=1}^{h+1} \mu_i(\tau_k) H_i$, and $H_i =

$$
\begin{bmatrix}
A + BD_{ci}C & BC_{ci} & \Gamma_{1i} + BK_{1i} & \Omega_i \\
B_{ci} & A_{ci} & F_{1i} & F_{0i} - B_{ci}CB \\
0 & 0 & 0 & I \\
D_{ci} & C_{ci} & K_{1i} & K_{0i} - D_{ci}CB
\end{bmatrix}
$$

with $\Omega_i = \Gamma_{0i} + BK_{0i} - (A + BD_{ci}C)B$.

3 Preliminary Results

Let us associate to the closed-loop system (6) a candidate parameter dependent Lyapunov function:

$$
V(z_k, \tau_k) = z_k^T Q^{-1}(\tau_k) z_k
$$

(7)

with $Q(\tau_k) = \sum_{i=1}^{h+1} \mu_i(\tau_k) Q_i$, $Q_i = Q_i^T > 0$.

By definition, the closed-loop system is robustly asymptotically stable, with contractivity coefficient $\lambda \in (0, 1]$ (or, robustly $\lambda$-contractive, for short), if:

$$
\Delta V(z_k, \tau_k) = V(z_{k+1}, \tau_{k+1}) - \lambda V(z_k, \tau_k) < 0
$$

(8)

$\forall z_k \in \mathbb{R}^l$, $z_k \neq 0$, and $\forall \tau_k \in [\tau_{\text{min}}, \tau_{\text{max}}]$.

When applied to (6), the above relation leads to the following inequality $^1$:

$$
(H + E\Delta D)^T(Q^+)^{-1}(H + E\Delta D) - \lambda Q^{-1} < 0
$$

(9)

Lemma 1 Let $\lambda \in (0, 1]$ be given and $\gamma$ computed from (4). The closed-loop system (6) is robustly $\lambda$-contractive, if there exist symmetric positive definite matrices $Q_i \in \mathbb{R}^{l_i \times l_i}$, matrix $U \in \mathbb{R}^{x \times l}$ and a scalar $\sigma > 0$ that verify, for $i, j = 1, ..., h + 1$:

$$
\begin{bmatrix}
-Q_j & HU & 0 & \gamma \sigma E \\
* & \lambda(Q_i - U^T U') & U'D' & 0 \\
* & * & -\sigma I & 0 \\
* & * & * & -\sigma I
\end{bmatrix} < 0
$$

(10)

Proof: By appropriately performing convex combinations of (10), first for $i$ and after for $j$ and applying the Schur complement twice, equation (10) leads to the following inequality, that depends on the parameter $r_k$:

$$
\begin{bmatrix}
-Q^+ + \sigma \gamma E' E' \\
U'H' & \lambda(Q - U - U') + \sigma^{-1}U'D'DU
\end{bmatrix} < 0
$$

This is equivalent to

$$
\begin{bmatrix}
-Q^+ \\
U'H' & \lambda(Q - U - U')
\end{bmatrix} +
\begin{bmatrix}
H \gamma E' & \gamma \Delta [0 \ DU] < 0
\end{bmatrix}
$$

Since $\Delta \Delta \leq \gamma^2 I$, the equation (11) is equivalent to (Petersen, 1987):

$$
\begin{bmatrix}
-Q^+ \\
U'H' & \lambda(Q - U - U')
\end{bmatrix} +
\begin{bmatrix}
H \gamma E' & \gamma \Delta [0 \ DU] < 0
\end{bmatrix}
$$

By the fact that $-U'Q^{-1}U \leq Q - U - U'$, the previous inequality implies

$$
\begin{bmatrix}
-(H + E\Delta D)^T & (H + E\Delta D) \\
\lambda Q^{-1}
\end{bmatrix} < 0
$$

that leads to (9). \quad \square

4 Dynamic output-feedback compensator

Inspired in the approach adopted in Castelan et al. (2010), we define the matrices $U$ and $U^{-1}$, the same used in Moraes et al. (2011):

$$
U = \begin{bmatrix}
X & \bullet & 0 \\
Z & \bullet & 0 \\
\Pi_1 & \bullet & 0 \\
\Pi_3 & \bullet & 0 \\
\end{bmatrix},
U^{-1} = \begin{bmatrix}
Y & \bullet & 0 \\
W & \bullet & 0 \\
\Sigma_1 & \bullet & 0 \\
\Sigma_3 & \bullet & 0 \\
\end{bmatrix}
$$

Furthermore, we have the following auxiliary matrix $\Theta$:

$$
\Theta = \begin{bmatrix}
Y & I & 0 \\
W & 0 & 0 \\
\Sigma_1 & 0 & I \\
\Sigma_3 & 0 & 0 \\
\end{bmatrix}
$$

$^1$For simplicity, the dependency on $r_k$ is omitted in the sequel, and the terms depending on $r_{k+1}$ are denoted with superscript $^+$. 

\[
\begin{bmatrix}
-\hat{Q}_j & M_{12i} & 0 & \gamma M_{14} \\
\ast & \lambda (\hat{Q}_j - \hat{U} - \hat{U}') & M_{23} & 0 \\
\ast & \ast & -\sigma I & 0 \\
\ast & \ast & \ast & -\sigma I \\
\end{bmatrix} < 0, \quad \forall i, j = 1, \ldots, h + 1
\]

where \( M_{14} = [\sigma Y \quad \sigma I \quad 0 \quad 0] \), \( M_{23} = [0 \quad -\Pi_1 + \Pi_3 \quad -I \quad I] \), and

\[
M_{12i} = \begin{bmatrix}
Y' A + \hat{B}_i C & A + B \hat{D}_i C & AX + B\hat{C}_i + \Gamma_i \Pi_3 + (\Gamma_{0i} - AB) \Pi_3 & \Gamma_{1i} + BK_{1i} \\
0 & \hat{A}_i & \Pi_3 & K_{1i} \\
\hat{D}_i C & \hat{C}_i & K_{0i} - \hat{D}_i CB & \end{bmatrix}
\]

It verifies:

\[
U\Theta = \begin{bmatrix}
I & X & 0 & 0 \\
0 & Z & 0 & 0 \\
0 & \Pi_1 & I & 0 \\
0 & \Pi_3 & 0 & I \\
\end{bmatrix}, \quad \tilde{U} = \begin{bmatrix}
Y' T' \Sigma_1' & \Sigma_3' \\
0 & \Pi_1' & I & 0 \\
0 & \Pi_3' & 0 & I \\
\end{bmatrix}
\]

where \( \tilde{U} = \Theta' U \Theta \) and, by construction:

\[T' = Y' X + W' Z + \Sigma_1' \Pi_1 + \Sigma_3' \Pi_3.\]

Moreover, we use the relation \( \bar{Q} = \Theta' Q \Theta \).

**Proposition 2** Consider a given scalar \( \lambda \in (0, 1] \), and compute \( \gamma \) from (4). If there exist symmetric positive definite matrices \( \hat{Q}_i \), matrices \( Y, X, T, \Sigma_1, \Sigma_3, \Pi_1, \Pi_3, A_i, B_i, F_{1i}, F_{0i}, \hat{C}_i, \hat{D}_i, K_{1i}, K_{0i} \) and a scalar \( \alpha > 0 \) that verify (12), then the \( n \)-order DOFC (5) with

\[
\begin{align*}
D_{ci} &= \hat{D}_i \\
C_{ci} &= (\hat{C}_i - \hat{K}_i \Pi_1 - (K_{0i} - D_{ci} CB) \Pi_3 - D_{ci} CX) Z^{-1} \\
B_{ci} &= (W')^{-1}(\hat{B}_i - Y BD_{ci} - \Sigma_1 D_{ci}) \\
F_{1ci} &= (W')^{-1}(\hat{F}_{1i} - Y' (\Gamma_{1i} + BK_{1i}) - \Sigma_1 K_{1i}) \\
F_{0ci} &= B_{ci} CB + (W')^{-1}(\hat{F}_{0i} - \Sigma_1 - \Sigma_3 (K_{0i} - D_{ci} CB)) \\
&\quad - Y'(\hat{F}_{0i} - \Sigma_3 (K_{0i} - D_{ci} CB)) \\
A_{ci} &= (W')^{-1}\left( \begin{array}{c}
\hat{A}_i - (W'F_{0ci} - B_{ci} CB) \\
Y' (\hat{F}_{0ci} + BK_{0i} - (A + BD_{ci} C)B) \\
\Sigma_1 (K_{0i} - D_{ci} CB) + \Sigma_3 D_{ci} \\
W' F_{1ci} + Y' (\hat{F}_{1ci} + BK_{1i}) + \Sigma_1 K_{1i} \Pi_1 \\
-(Y' A + W' B_{ci} C) X \end{array} \right) Z^{-1} - \Sigma_3 C_{ci} - Y' B_{ci}
\end{align*}
\]

is such that (6) is robustly \( \lambda \)-contractive.

**Proof:** By pre- and post-multiplying (10) by \( \text{diag}\{\Theta', \Theta', I, I\} \) and its transpose, respectively, and defining the auxiliary variables

\[
\begin{align*}
\hat{D}_i &= D_{ci} \\
\hat{C}_i &= C_{ci} + Z + K_{0i} \Pi_1 + (K_{0i} - D_{ci} CB) \Pi_3 + D_{ci} CX \\
\hat{B}_i &= Y' BD_{ci} + W' B_{ci} + \Sigma_3 D_{ci} \\
\hat{F}_{1i} &= W' F_{1i} + Y' (\hat{F}_{1ci} + BK_{1i}) + \Sigma_1 K_{1i} \\
\hat{F}_{0i} &= \Sigma_1 + W' (F_{0ci} - B_{ci} CB) + \Sigma_3 (K_{0i} - D_{ci} CB) \\
&\quad + Y' (\hat{F}_{0i} - \Sigma_3 (K_{0i} - D_{ci} CB)) \\
\hat{A}_i &= \hat{F}_{0i} \Pi_3 + \hat{F}_{1i} \Pi_3 + (Y' A + \hat{B}_i C) X \\
&\quad (W' A_{ci} + \Sigma_3 C_{ci} + Y' B_{ci}) Z
\end{align*}
\]

we obtain the equivalence between conditions (10) and (12).

Observe that the condition presented in Proposition 2 is a BMI with respect to some decision variables, i.e., the matrix \( M_{14} \) is a non-linear function of variables \( \sigma \) and \( Y \). Such a condition can be solved, for instance, by fixing a value to \( \sigma \) and considering \( Y \) as a decision variable. However, the choice of \( \sigma \) is not trivial.

A purely LMI condition might be proposed by considering \( D_{ci}(\tau_k) = 0 \) \( \forall k \), which results in a particular version of compensator (5).

**Corollary 3** Consider a given scalar \( \lambda \in (0, 1] \), and compute \( \gamma \) from (4). If there exist symmetric positive definite matrices \( \hat{Q}_i \), matrices \( Y, X, T, \Sigma_1, \Sigma_3, \Pi_1, \Pi_3, A_i, B_i, F_{1i}, F_{0i}, \hat{C}_i, \hat{D}_i, K_{1i}, K_{0i} \) and a scalar \( \alpha > 0 \) that verify (13), then the \( n \)-order DOFC (5) with

\[
\begin{align*}
C_{ci} &= (\hat{C}_i - \hat{K}_i \Pi_1 - K_{0i} \Pi_3) Z^{-1} \\
B_{ci} &= (W')^{-1}\hat{B}_i \\
F_{1ci} &= (W')^{-1}(\hat{F}_{1i} - Y' (\Gamma_{1i} + BK_{1i}) - \Sigma_1 K_{1i}) \\
F_{0ci} &= B_{ci} CB + (W')^{-1}(\hat{F}_{0i} - \Sigma_1 - \Sigma_3 K_{0i}) \\
&\quad - Y'(\hat{F}_{0i} - \Sigma_3 K_{0i} - AB) \\
A_{ci} &= (W')^{-1}\left( \begin{array}{c}
\hat{A}_i - (W'F_{0ci} - B_{ci} CB) \\
Y' (\hat{F}_{0ci} + BK_{0i} - (A + BD_{ci} C)B) \\
\Sigma_1 (K_{0i} - D_{ci} CB) + \Sigma_3 D_{ci} \\
W' F_{1ci} + Y' (\hat{F}_{1ci} + BK_{1i}) + \Sigma_1 K_{1i} \Pi_1 \\
-(Y' A + W' B_{ci} C) X \end{array} \right) Z^{-1} - \Sigma_3 C_{ci} - Y' B_{ci}
\end{align*}
\]

is such that (6) is robustly \( \lambda \)-contractive.

**Proof:** By pre- and post-multiplying (12) by \( \text{diag}\{\Phi, \Phi, \frac{1}{\sqrt{m_i}}, \frac{1}{\sqrt{m_i}}, \frac{1}{\sqrt{m_i}}, \frac{1}{\sqrt{m_i}}\} \), with \( \Phi = \text{diag}\{\sqrt{m_i}, \frac{1}{\sqrt{m_i}}, \frac{1}{\sqrt{m_i}}, \frac{1}{\sqrt{m_i}}, \frac{1}{\sqrt{m_i}}, \frac{1}{\sqrt{m_i}}\} \), and by considering \( D_{ci} = 0 \) \( \forall i = 1, \ldots, h + 1 \), we obtain (13) with \( \alpha = \frac{1}{\sqrt{m_i}}, \hat{B}_i = \sigma \hat{B}_i, \hat{C}_i = \sigma \hat{C}_i, \hat{K}_{1i} = \sigma K_{1i}, \hat{K}_{0i} = \sigma K_{0i}, \hat{Y} = \sigma Y, \hat{X} = \sigma X, \hat{\Pi}_1 = \alpha \hat{\Pi}_1, \hat{\Pi}_3 = \alpha \hat{\Pi}_3, \) and

\[
\tilde{U} = \begin{bmatrix}
Y' & T' & \Sigma_1' & \Sigma_3' \\
I & X & 0 & 0 \\
0 & \Pi_1 & \alpha I & 0 \\
0 & \Pi_3 & 0 & \alpha I \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
-\tilde{Q}_j & \tilde{M}_{12i} & 0 & \gamma \tilde{M}_{14} \\
* & \lambda (\tilde{Q}_i - \tilde{U} - \tilde{U}') & \tilde{M}_{23} & 0 \\
* & * & -I & 0 \\
* & * & * & -I
\end{bmatrix} < 0, \quad \forall i, j = 1, ..., h + 1
\]  

where \( \tilde{M}'_{14} = [\tilde{Y} \ I \ 0 \ 0] \), \( \tilde{M}'_{23} = [0 \ -\tilde{\Pi}_1 + \tilde{\Pi}_3 - \alpha I \ \alpha I] \), and
\[
\tilde{M}_{12i} = \begin{bmatrix}
\tilde{Y}'A + \tilde{B}_iC & A & \tilde{A}_i & \tilde{A}_i \\
0 & \tilde{X} + \tilde{B}_iC + \Gamma_i\tilde{\Pi}_1 + (\Gamma_{0i} - AB)\tilde{\Pi}_3 & \alpha \Gamma_{1i} + B\tilde{K}_{1i} & \alpha (\Gamma_{0i} - AB) + B\tilde{K}_{0i} \\
0 & \tilde{C}_i & 0 & \tilde{K}_{1i} \\
0 & 0 & 0 & \tilde{K}_{0i}
\end{bmatrix}
\]

5 Example and Simulation

Consider the double integrator system (see Cloosterman et al. (2010), Moraes et al. (2011)):
\[
M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

The sampling period is \( T = 0.1s \) and the delay bounds are \( \tau_{\text{min}} = 0.001s \) and \( \tau_{\text{max}} = T \). The following results were obtained using: Yalmip, Sedumi and True-time.

The feasibility of LMIs (13) has been verified in order to find the smallest values of the contractivity coefficient, \( \lambda_{sm} \). In order to achieve that, a linear search from \( \lambda = 1 \) until reaching \( \lambda = \lambda_{sm} \) has been made. Recall that, the contractivity coefficient \( \lambda \) is related to the rate of the convergence of the trajectories to the origin.

Using the linear approach, defined by Corollary 3 (C3), we managed to obtain a contractive coefficient \( \lambda_{smC3} = 0.8017 \), with corresponding value of \( \sigma = 43.5907 \). This same value of \( \sigma \) has been used to determine contractive coefficients for the nonlinear approaches defined by Proposition 2 (P2) and by an adapted version of the compensator presented in Moraes et al. (2011). For the first case, we obtained \( \lambda_{smP2} = 0.7976 \), while for the second case we obtained \( \lambda_{smSBAI'11} = 0.7990 \). Observe that, in the three cases, we obtained similar contractivity coefficients. Still, these results are merely demonstrative, once there might be another \( \sigma \in R \), for the nonlinear approach, that provides some lower value for \( \lambda = \lambda_{sm} \), but the search for this value is not trivial.

For the simulations shown in Fig. 2 and Fig. 3, the compensators calculated by the use of Corollary 3 and by the use of Proposition 2, with each respective \( \lambda_{sm} \), were used. The corresponding DOFC matrices are shown in Table 1. The initial condition was \( x_0 = [3 - 1] \) for both cases.

It is important to emphasize that the same delay sequence has been used to simulate the behavior of the NCS with the two computed DOFCs. The variable delay was generated by an uniform distribution, using the limits \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \).

As might be expected, due to the values of \( \lambda_{sm} \) being close to each other, the output behavior is similar in both cases, as shown in Fig. 2. A more noticeable difference occurs at the initial response of the control efforts, where the compensator with \( D_c = 0 \) has a slight delay, corresponding to one discrete-time period \( k \), compared to the compensator calculated by Proposition 2. The control signals are shown in Fig. 3.

6 Conclusion

We presented some results concerning the synthesis of a dynamic output feedback compensator for NCS. The proposed compensator has the same order of the plant and its structure depends on the delay induced by the network. Thus, the induced delay must be considered real-time available, through time-stamped messages for example. The proposed results are complementary to previous results proposed in Moraes et al. (2011). A more general structure of DOFC, in which the \( n \)-order compensator can be viewed as a particular case, can also be proposed, and will be the subject of a future publication. Furthermore, these techniques can be adapted to deal with estimated delays.
Table 1: Numerical results for $h = 2$

<table>
<thead>
<tr>
<th>$A_{i}$</th>
<th>$0.0656$</th>
<th>$0.1075$</th>
<th>$0.0722$</th>
<th>$0.1066$</th>
<th>$0.0415$</th>
<th>$0.0699$</th>
<th>$0.4112$</th>
<th>$0.0689$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{i}$</td>
<td>$0.7145$</td>
<td>$0.4778$</td>
<td>$-0.7185$</td>
<td>$0.4636$</td>
<td>$0.2688$</td>
<td>$0.0442$</td>
<td>$0.2640$</td>
<td>$0.0445$</td>
</tr>
<tr>
<td>$C_{i}$</td>
<td>$0.0742$</td>
<td>$0.1066$</td>
<td>$-0.7197$</td>
<td>$0.4624$</td>
<td>$0.2637$</td>
<td>$0.0445$</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$D_{i}$</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>$0.2907$</td>
<td>$0.0479$</td>
<td>$0.2968$</td>
<td>$0.0489$</td>
</tr>
</tbody>
</table>

References


